Chiral Symmetry and the Low-Energy Spectrum of the QCD Dirac Operator

J.J.M. Verbaarschot,
Department of Physics and Astronomy,
SUNY at Stony Brook, Stony Brook, NY 11794

The order parameter of the chiral phase transition is directly related to the infrared part of the spectrum of the QCD Dirac operator. This part of the spectrum follows from the low energy limit of QCD which is given by a partition function of weakly interacting Goldstone modes. We find that the slope of the Dirac spectrum is determined by the pion decay constant whereas for $\lambda \ll 1/L^2\Lambda_{\rm QCD}$ the correlations of the Dirac eigenvalues are given by a random matrix theory with the global symmetries of the QCD partition function. A possible observation of these continuum results in lattice QCD with staggered fermions is discussed.

1. Introduction

It has been well established that QCD is the correct theory of the strong interactions. The vast body of evidence from lattice QCD simulations at low and intermediate energies [1] is complemented by perturbative calculations which become reliable at high energies where the renormalized coupling constant is small due to asymptotic freedom. In this lecture we will focus on another limit where QCD simplifies. Because of spontaneous breaking of chiral symmetry, QCD at low energy reduces to a theory of weakly interacting Goldstone bosons. Although this theory cannot be derived from QCD by means of an ab-initio calculation, its Lagrangian is determined uniquely by chiral symmetry and Lorentz invariance. The validity of this low-energy theory is based on the presence of a mass-gap which is a highly nontrivial and nonperturbative feature of QCD.

One of the questions we wish to address is to what extent the existence of an effective low-energy theory imposes consistency conditions on the original theory. This question was first posed in [2] for the mass dependence of the QCD partition function. They argued that for small quark masses it can be both obtained from the effective partition function and from QCD. Since the mass dependence of the QCD partition function is given by the average of the fermion determinant this imposes consistency conditions on the average properties of the eigenvalues of the QCD Dirac operator.

Some of the properties of the Dirac eigenvalues are robust against *large* deformations of the gauge field action well outside the scaling window. This type of spectral universality has been investigated primarily within the context of Random Matrix Theory [3,4]. What has been found is that correlations of eigenvalues on the scale of the average level spacing are universal, i.e. they are robust against deformations of the probability distribution of the matrix elements. The low-energy effective action is also insensitive to *large* deformations of the gauge field action. The reason is the existence of a mass gap. In the next section we will relate this property to spectral universality.

2. Valence Quarks and the Low-Energy Limit of QCD

Because the same mass occurs both in the quark propagator and in the fermion determinant the average Dirac spectrum cannot be obtained directly from the QCD partition function. In order to access the spectrum of the Euclidean Dirac operator D one has to introduce the valence quark mass dependence of the chiral condensate defined by [5–7]

$$\Sigma(m_v) = \langle \mathrm{Tr} \frac{1}{m_v + D} \rangle. \tag{1}$$

The average is over the Euclidean QCD action which includes a fermion determinant that depends on the the sea-quark masses. The spectral density per unit space-time volume $V = L^4$ of the Dirac operator D is directly related to to $\Sigma(m_v)$,

$$\rho(\lambda)/V = \frac{1}{2\pi} \left(\Sigma(i\lambda + \epsilon) - \Sigma(i\lambda - \epsilon) \right). \tag{2}$$

The generating function for $\Sigma(m_v)$ is defined by [8,7,9]

$$Z^{\mathrm{pq}}(m_v, J) \;=\; \int [dA] \, \frac{\det(D + m_v + J)}{\det(D + m_v)} \prod_{f=1}^{N_f} \det(D + m_f) \; e^{-S_{YM}[A]} \;. \tag{3}$$

In addition to the usual quarks this partition function contains valence quarks and its bosonic superpartners. The chiral condensates for the different (super-)flavors are given by the same expression in terms of the eigenvalues of the Dirac operator. We thus have a maximum breaking of the axial flavor symmetry. The low-energy modes are then given by the Goldstone modes associated with the spontaneous breaking of this symmetry. Their quark content can be either two sea-quarks, one sea quark and one valence quark, or two valence quarks. The low-energy effective partition follows from the flavor supersymmetry and its breaking and Lorentz invariance as is the case for the usual chiral Lagrangian [8,7,9]. One major difference is that in this case the Goldstone manifold is a super-manifold with both compact and non-compact degrees of freedom [10,7,9].

3. Scales in the Dirac Spectrum

For a nonzero value of the chiral condensate Σ we can identify three important scales in the Dirac spectrum. The first scale is the smallest nonzero eigenvalue of the Dirac operator given by $\lambda_{\mathrm{min}} = 1/\rho(0) = \pi/\Sigma V$. The second scale is the valence quark mass for which the Compton wavelength of the associated Goldstone bosons is equal to the size of the box. Using the Gell-Mann-Oakes-Renner relation we obtain as Thouless energy[11,12,5,6]

$$m_c = \frac{F^2}{\Sigma L^2}, \tag{4}$$

where F the pion decay constant. A third scale is given by a typical hadronic mass scale. The three scales are ordered as $\lambda_{\mathrm{min}} \ll m_c \ll \Lambda$. For valence quark masses $m_v \ll m_c$

the kinetic term in the effective action can be neglected and the low-energy partition function can be reduced to a zero dimensional integral. However, any partition function with a mass gap and the same pattern of chiral symmetry breaking as in QCD can be reduced this way. The simplest such theory is chiral Random Matrix Theory (chRMT) [14]. In that case spontaneous breaking of chiral symmetry arises in the limit of infinite matrices. The advantage of working with chRMT is that is relatively simple to derive the distribution of the smallest eigenvalues. Of course, the results for $\Sigma(m_v)$ obtained from the partially quenched partition function and from chRMT coincide [7,9].

The kinetic term is also determined uniquely by chiral symmetry and Lorentz invariance which allows us to calculate the Dirac spectrum in the domain $m_v \ll \Lambda$. This results in the slope of the Dirac spectrum at $\lambda = 0$ which for N_f massless flavors is given by [7,13]

$$\frac{\rho'(0)}{\rho(0)} = \frac{(N_f - 2)(N_f + \beta)}{16\pi\beta} \frac{\Sigma_0}{F^4}. \tag{5}$$

Here, β denotes the Dyson index of the Dirac operator. For QCD with fundamental fermions and three or more colors (with $\beta = 2$) this result was first derived in [15]. The other two possibilities, $\beta = 1$ and $\beta = 4$, refer to QCD with fundamental fermions and two colors and QCD with adjoint fermions and two or more colors, respectively.

The domain below m_c has been investigated extensively by means of lattice QCD simulations and agreement with the chRMT results has been found [5,16–23]. A somewhat surprising result is that the lattice QCD data reproduce the analytical result for zero topological charge. This will be explained in the next section.

4. Approach to the Continuum Limit for Staggered Fermions

The low-energy limit of QCD and the small Dirac eigenvalues are described by the same partition function. In order to recover the continuum $U_A(1)$ symmetry of the staggered Dirac operator (without the $U_A(1)$ symmetry) its smallest eigenvalues thus have to approach their continuum limit as well. The staggered Dirac operator can be written as

$$D_{KS} = D_C + a^2\Lambda^2 D_R, \tag{6}$$

where D_C coincides with the continuum Dirac operator at low energies, a is the lattice spacing and Λ is a typical hadronic mass scale. The condition that the Dirac spectrum of D_{KS} approaches that of D_C is (with $||\cdot||$ the norm of an operator)

$$||a^2\Lambda^2 D_R|| \ll \Delta\lambda a. \tag{7}$$

With $\Sigma \sim \lambda^3$, the spacing of the eigenvalues near zero in lattice units is given by $\Delta\lambda a \sim 1/\rho(0) \sim 1/N\Lambda^{d-1}a^{d-1}$. Since $||D_R|| \sim O(1)$, the condition (7) can be rewritten as

$$a^{d+1}\Lambda^{d+1} \ll \frac{1}{N} \qquad \text{or} \qquad L\Lambda = N^{1/d}a\Lambda \ll N^{\frac{1}{d}-\frac{1}{d+1}}. \tag{8}$$

But we also require a sufficiently large lattice with $L\Lambda \gg 1$ resulting in

$$N^{\frac{1}{d(d+1)}} \gg 1. \tag{9}$$

Our *naive* estimate for the total number of lattice points for staggered fermions to approach the continuum limit is given by $N \approx (3^{d+1})^d$. In two dimensions we need lattices of the order of 27^2 to be reasonably close to the continuum limit. This number is consistent with simulations of the Schwinger model with staggered fermions [17]. In four dimensions the situation is much worse. According to the same estimate continuum physics is only seen on lattices as large as 343^4 which explains that todays staggered lattices show agreement with chRMT results in the sector of zero topological charge [16,18,23,24].

5. Conclusions

We have argued that the the distribution of the smallest eigenvalues of the Dirac operator is a signature for the pattern of chiral symmetry breaking of the QCD partition function. Both the correlations of the smallest eigenvalues and the slope of the Dirac spectrum from have been obtained from a chiral Lagrangian. The intercept of the Dirac spectrum determines the chiral condensate whereas its slope fixes the pion decay constant. However, it takes very large staggered lattices to reliably perform such analysis.

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REFERENCES

1. C. DeTar, *Quark-gluon plasma in numerical simulations of QCD*, in *Quark gluon plasma 2*, R. Hwa ed., World Scientific 1995.
2. H. Leutwyler and A. Smilga, Phys. Rev. **D46** (1992) 5607.
3. G. Hackenbroich and H.A. Weidenmüller, Phys. Rev. Lett. **74** (1995) 4118.
4. G. Akemann, P. Damgaard, U. Magnea and S. Nishigaki, Nucl. Phys. **B 487[FS]** (1997) 721.
5. J.J.M. Verbaarschot, Phys. Lett. **B368** (1996) 137.
6. J.C. Osborn and J.J.M. Verbaarschot, Phys. Rev. Lett. **81** (1998) 268.
7. J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nuc. Phys. **B540** (1999) 317.
8. C. Bernard and M. Golterman, Phys. Rev. D49 (1994) 486; C. Bernard and M. Golterman, hep-lat/9311070; M. Golterman and K.C. Leung, Phys. Rev. **D57** (1998) 5703.
9. P.H. Damgaard, J.C. Osborn, D. Toublan and J.J.M. Verbaarschot, Nucl. Phys. **B547** (1999) 305.
10. M. Zirnbauer, J. Math. Phys. **37** (1996) 4986.
11. B.L. Altshuler, I.Kh. Zharekeshev, S.A. Kotochigova and B.I. Shklovskii, Zh. Eksp. Teor. Fiz. **94** (1988) 343.
12. J. Gasser and H. Leutwyler, Phys. Lett. **188B** (1987) 477.
13. D. Toublan and J.J.M. Verbaarschot, hep-th/9904199.
14. E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. **A560** (1993) 306; J.J.M. Verbaarschot, Phys. Rev. Lett. **72** (1994) 2531.
15. A. Smilga and J. Stern, Phys. Lett. **B318** (1993) 531.
16. M.E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J.J.M. Verbaarschot and T. Wettig, Phys. Rev. Lett. **80** (1998) 1146.

17. F. Farchioni, I. Hip, C.B. Lang, M. Wohlgenannt, hep-lat/9812018.
18. R.G. Edwards, U.M. Heller and R. Narayanan, hep-lat/9902021.
19. F. Farchioni, I. Hip and C.B. Lang, hep-lat/9907011.
20. M.E. Berbenni-Bitsch, M. Gockeler, H. Hehl, S. Meyer, P.E. Rakow, A. Schafer and T. Wettig, hep-lat/9907014.
21. P.H. Damgaard, R.G. Edwards, U.M. Heller and R. Narayanan, hep-lat/9907016.
22. P. Hernandez, K. Jansen and L. Lellouch, hep-lat/9907022.
23. P.H. Damgaard, U.M. Heller, R. Niclasen and K. Rummukainen, hep-lat/9907019.
24. P. de Forcrand, Aspen workshop on the Dirac Spectrum.